§5. CONFLICT MODELS

One model that I did not consider in discussing Romeo and Juliet was

$$\frac{dR}{dt} = -a\,J\ , \tag{5.1a}$$

$$\frac{dJ}{dt} = -b\,R\ . \tag{5.1b}$$

Why not? In this scenario, Romeo and Juliet respond negatively to each other. This is hardly the stuff of love. If anything, it is the stuff of animosity, hate, and aggression.

Having said this, I should now point out that simple systems of differential equations have also been used to model conflict and aggression. Indeed there is a surprisingly long history to these models. There are two famous classes of conflict models:

(a) Lanchester Models (Combat or Attrition Models), and

(b) Richardson Models (Arms Race Models).

Frederick William Lanchester (1868 - 1945) was an eclectic English engineer who made contributions to automotive design and the theory of aerodynamics. Interestingly enough, he is best remembered for his equations of war, first set out in his 1916 book *Aircraft in Warfare: The Dawn of the Fourth Arm.* Lewis Frye Richardson, in turn, was a physicist of Quaker background whose aversion to war led him to undertake a study of its causes. His essay on the ‘‘Mathematics of war and foreign politics’’ was published in 1956.

I should add that recent historical research has revealed that M. Osipov, a Russian, developed a theory of attrition models at roughly the same time as Lanchester. In addition, both Lanchester and Osipov were preceded by more than a decade by J. V. Chase, an American naval lieutenant. Their work has only come to light recently due to the difficulties of international scientific communication (in the case of Osipov) and because of the Naval classification system (in the case of Chase).

I would now like to spend a little bit of time discussing some Lanchester models; they are closely related to the systems that we have been discussing. Last week, we were paying homage to ‘‘Shakespeare in Love’’. This week, I guess, we’re paying homage to ‘‘Saving Private Ryan’’.

First of all, let me say a few words about what we can expect from a mathematical theory of combat. Warfare is quite complex and the outcome of any particular battle may depend on factors that are extremely hard to measure before the fact. Theoreticians typically seek simple models that generate a reasonable set of *core qualitative behaviors.* Having said that, we will also see that models can give reasonable agreement to some historical data sets. Let me go ahead then and discuss Lanchester models.

In a Lanchester model, an x force and a y force are engaged in battle. The variables $x(t)$ and $y(t)$ denote the strength of the forces at time t, where t is measured in days since the start of the conflict. Needless to say, it is not always easy to quantify the ‘‘strength’’ of a force. This may depend upon training, leadership, morale, and many other factors. We’ll take the easy way out and identify the strength with the number of combatants or soldiers. The variables x and y must, of course, be nonnegative. We shall assume that $x(t)$ and $y(t)$ are also continuous functions of time. This is, of course, an idealization since the real number of soldiers is, in reality, a discrete integer.

It is difficult to write a specific formula for $x(t)$ *a priori* . We may, nevertheless, write down an equation for the rate at which soldiers are added or lost. A Lanchester model typically assumes that, for force x,

$$\frac{dx}{dt} = -(\mathrm{OLR} + \mathrm{CLR}) + \mathrm{RR} . \tag{5.2}$$

OLR is the **operational loss rate**, the loss rate due to diseases, desertions, and other noncombat mishaps. CLR is the **combat loss rate**, and RR is the **reinforcement rate**. A similar equation applies to y. I’m not particularly fond of acronyms such as OLR, but the military is the most prolific source of these nonwords, and so it is probably appropriate that they appear frequently in combat models.

Let me outline three, very traditional Lanchester combat models.

(a) Conventional Combat (CONCOM)

$$\frac{dx}{dt} = -a\,x - b\,y + P(t) , \tag{5.3a}$$

$$\frac{dy}{dt} = -c\,x - d\,y + Q(t) .$$

(b) Guerilla Combat (GUERCOM)

$$\frac{dx}{dt} = -a\,x - g\,x\,y + P(t) , \tag{5.4a}$$

$$\frac{dy}{dt} = -c\,x - h\,x\,y + Q(t) .$$

(c) Mixed Guerilla–Conventional Combat (VIETNAM)

$$\frac{dx}{dt} = -a\,x - g\,x\,y + P(t)\;, \tag{5.5a}$$

$$\frac{dy}{dt} = -c\,x - d\,y + Q(t)\;.$$

Each model has operational loss rates that are proportional to the number of troops. For each force, in other words, there is a constant per soldier operational loss rate. It is in the combat loss rates that the models differ. These terms reflect the interaction between the two forces.

A conventional force typically operates in the open. Lanchester assumed that every member of a conventional force is within range of the enemy and that conventional force x has a loss rate, $-by$, that is proportional to the number of enemy troops. In effect, it is the firepower available to the enemy that matters. The proportionality constant b is referred to as the *combat effectiveness coefficient*. This parameter is difficult to measure. One approach is to set

$$b = r_y\,p_y\;, \tag{5.6}$$

where r_y is the **firing rate** (shots/combatant/day) of the y force and p_y is the **probability that a single shot kills an opponent**. The parameter b is typically easier to analyze after the fact than beforehand. Similar arguments apply for the term $-cx$.

I've argued that the loss rate for a conventional force is linear. The combat loss rate for a guerilla force is generally thought to be nonlinear. Imagine that an invisible guerilla force occupies some region R. The enemy fires blindly into R. Under these circumstances, we may imagine that the loss rate for the guerillas is proportional to their number $x(t)$ in region R. The greater $x(t)$, the greater the probability that an enemy's shot will be effective. Thus we have that the combat loss rate for the guerillas is

$$CLR = g\,x(t)\,y(t)\;. \tag{5.7}$$

I've described the above nonlinear form as the combat loss rate for guerillas. It has also been used to model two other, seemingly contradictory scenarios. Lanchester originally introduced this form to describe loss rates in ancient combat. He emphasized the idea that ancient combat was one-on-one and that ancient forces could not concentrate their firepower. The same formulation has also been used as an area-fire model for artillery duels and other situations involving indirect, blind, or inefficient firing.

Determining g is *extremely* difficult. In the context of guerilla warfare, the combat effectiveness coefficient g is often assumed to be proportional to the **area of effectiveness** A_{ry} of a single y shot, and inversely proportional to the area A_x occupied by the guerillas,

$$g = r_y \frac{A_{ry}}{A_x} . \tag{5.8}$$

(The number A_{ry} is often taken as the area of the exposed part of the body of a single gorilla combatant under cover.) In other words, the guerillas will suffer lower losses if they spread themselves thin.

Before looking at a complicated scenario, let's look at several simplified versions of CONCOM, GUERCOM, and VIETNAM.

Example: Simplified CONCOM — The Square Law

Let's consider the case of two conventional armies with no operational loss terms and no replacements,

$$\frac{dx}{dt} = -b\,y , \tag{5.9a}$$

$$\frac{dy}{dt} = -c\,x . \tag{5.9b}$$

The attrition rate of each belligerent is proportional to the size of the adversary.

This is a linear system that we can analyze in the phase plane. I say the phase plane, even though the positive quadrant is all we care about. The matrix for the linearized system is

$$\mathbf{A} = \begin{pmatrix} 0 & -b \\ -c & 0 \end{pmatrix} \tag{5.10}$$

and the eigenvalues are clearly

$$\lambda = \pm\sqrt{bc} \tag{5.11}$$

so that we're dealing with a saddle point. I'll leave it to you to find the eigenvectors and to write out the complete solution. Nevertheless, it should be clear from equations (5.9a) and (5.9b) that both forces are decreasing in the first quadrant. From the same equations, we also have that

$$\frac{dy}{dx} = \frac{cx}{by} . \tag{5.12}$$

Separating the variables and integrating, we obtain

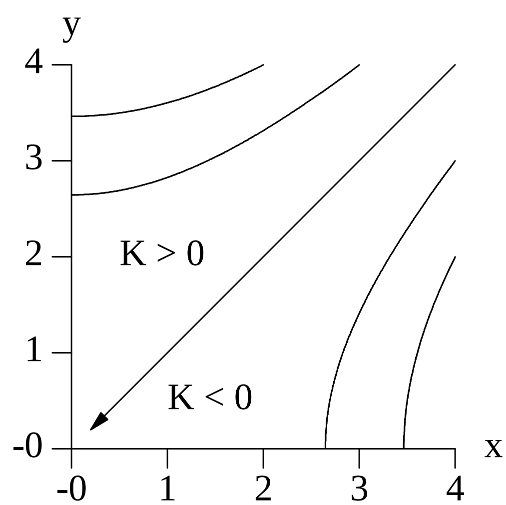
$$b \int_{y_o}^{y(t)} y\, dy \;=\; c \int_{x_o}^{x(t)} x\, dx \tag{5.13}$$

$$b\,[y^2(t) - y_0^2] \;=\; c\,[x^2(t) - x_0^2]\;. \tag{5.14}$$

This quadratic relationship is the source of the name ‘‘square law model’’. This quadratic relationship also implies that the orbits are hyperbolas, unless

$$K \;\equiv\; b\,y_0^2 - c\,x_0^2 \;=\; 0\;, \tag{5.15}$$

in which case we get straight lines. Some typical orbits are plotted in the next figure.



Square Law

Who wins this war? One can argue that x wins if y vanishes first, and vice versa. By this criterion, x wins if $K < 0$ and y wins if $K > 0$. A stalemate occurs if

$$b\,y_0^2 \;=\; c\,x_0^2\;, \tag{5.16}$$

or

$$y_0 \;=\; \sqrt{\frac{c}{b}}\; x_0\;. \tag{5.17}$$

This equation is very important. It says that to stalemate an adversary three times as numerous, it does not suffice to be three times as effective; you must be nine times as effective! This presumed heavy advantage in numbers is deeply embedded in most modern Pentagon models. It played a particularly important role during the cold war, when the Russians were seen as having a huge advantage numerical advantage in conventional weapons in Central Europe.

Example: Simplified GUERCOM — The Linear Law

Let's now look at two guerilla forces that have no operation losses and no reinforcements,

$$\frac{dx}{dt} = -g\,x\,y\,, \tag{5.18a}$$

$$\frac{dy}{dt} = -h\,x\,y\,. \tag{5.18b}$$

In this case,

$$\frac{dy}{dx} = \frac{h}{g} \tag{5.19}$$

so that

$$g\,[y(t) - y_0] = h\,[x(t) - x_0]\,. \tag{5.20}$$

In this case, stalemate is achieved when

$$L \equiv g\,y_0 - h\,x_0 = 0 \tag{5.21}$$

or

$$\frac{y_0}{x_0} = \frac{h}{g}\,. \tag{5.22}$$

In this case, the ratio of initial numbers is *not* magnified by squaring. If an opponent is three times as numerous, you must only be three times as effective.

For this model, numerical advantage is not nearly as critical as in conventional combat. You can begin to see why it is useful in dealing with guerilla warfare and ancient warfare. It is certainly the case that there are many examples in ancient history of numerically inferior forces defeating numerically superior forces. In the battle of Marathon 10,000 Athenians defeated a force of 100,000 Persians. The Greek loss was 192, the Persian loss was 6,400. (Let's not forget Thermopylae either.) The Bible describes the ‘‘War at Gibeah’’ between 400,000 (!) Israelite swordsmen and 26,000 Benjamite swordsmen in which the Israelites took huge losses in the first two days of battle. The Benjamites, however, were helped by the presence of 700 slingers — shades of a concentration of firepower.

Example: Simplified VIETNAM — The Parabolic Law

In VIETNAM, a guerilla force opposes a conventional force. Once again, we will ignore operational losses and reinforcements,

$$\frac{dx}{dt} = -g\,x\,y\;, \tag{5.23a}$$

$$\frac{dy}{dt} = -c\,x\;. \tag{5.23b}$$

Clearly

$$\frac{dy}{dx} = \frac{c}{gy}\;. \tag{5.24}$$

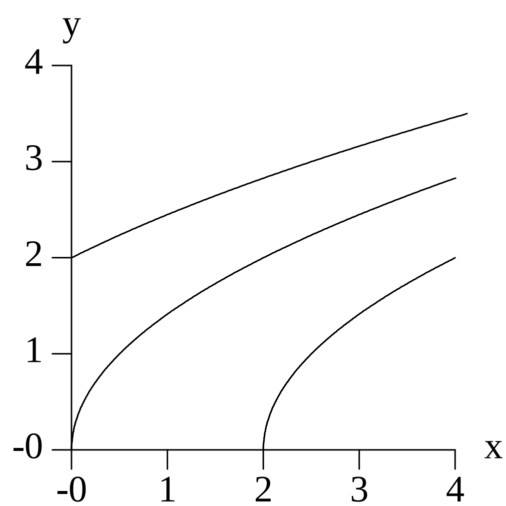
Integrating, we get that

$$g\,y^2(t) = 2\,c\,x(t) + M$$

where

$$M \equiv g\,y_0^2 - 2\,c\,x_0\;.$$

The guerilla force wins if $M < 0$, and the conventional force wins if $M > 0$.



Parabolic Law

In order for there to be a stalemate, we require that $M = 0$ or that

$$\left(\frac{y_0}{x_0}\right) > \sqrt{\frac{2c}{g\,x_0}}\;. \tag{5.25}$$

Since

$$c = r_x \, p_x \, , \tag{5.26}$$

and

$$g = r_y \frac{A_{ry}}{A_x} \, , \tag{5.27}$$

we require that

$$\left(\frac{y_0}{x_0}\right) > \sqrt{2 \, \frac{r_x}{r_y} \, \frac{A_x \, p_x}{A_{ry}} \, \frac{1}{x_0}} \, . \tag{5.28}$$

Let's do a quick estimate of this ratio. Let's assume that the firing rates are approximately equal. In addition, suppose that the probability that a shot by a guerilla kills an opponent is $p_x = 0.1$ and that the vulnerable part of the body of a single guerilla combatant, A_{ry}, is 2 sq. ft. Thus we have that

$$\left(\frac{y_0}{x_0}\right) > \sqrt{\frac{0.1 \, A_x}{x_0}} \, . \tag{5.29}$$

Now, guerilla forces usually operate in relatively small units, so let's set $x_0 = 100$ and assign 1000 sq. ft. to each guerilla combatant, so that

$$A_x = 100 \times 1000 = 100,000. \tag{5.30}$$

Under all these hypotheses, our ratio reduces to

$$\left(\frac{y_0}{x_0}\right) > \sqrt{\frac{0.1 \times 100,000}{100}} = 10 \, . \tag{5.31}$$

Thus the simplified VIETNAM model indicates that the ratio of conventional forces to guerilla forces must be quite large if the guerilla operate in small units in relatively large regions.

In 1962, Deitchman[1] listed the average force ratios for 10 mixed conventional–guerilla wars following World War II. The conventional forces typically won if their force ratio exceeded 8. Vietnam was not, of course, a purely conventional–guerilla conflict; the north sometimes fought as a conventional force. Nevertheless, it is interesting to note that the force ratio there never exceeded 6. It would be really interesting to go back and redo Deitchman's study using more recent conflicts also.

I want to conclude this section by examining at least one historical conflict. I went to the library the other day to see what examples were

[1] S. J. Deitchman, "A Lanchester model of guerilla warfare," *Operations Research*, 10, pp. 818-827 (1962).

available in the recent literature. Not all battles, of course, will do; you really need to have accurate day-by-day casualty figures. In spite of this, I did find a number of examples. These included models for the ‘‘War at Gibbeah’’,[2] for the Ardennes campaign[3, 4] (the Battle of the Bulge), and a reexamination[5] of an earlier model for the Inchon-Seoul campaign of the Korean War. Many early examples of the use Lanchester models occur in the journal *Operations Research*. This journal has more of a business bent these days, and most of military applications of Lanchester models now show up in the journal *Naval Research Logistics*. *Operations Research* does still carry some articles that use Lanchester models, but they seem to be mostly models for advertising wars. In spite of these recent interesting paper, I though it best to stay with a classic example of the use of a Lanchester model. This is Engel's[6] use of a Lanchester model to describe the Battle of Iwo Jima.

The Battle of Iwo Jima was one of the fiercest battles of the Second World War. I think most of you have seen the famous photograph of four American marines raising the U.S. flag on Iwo Jima's Mt. Suribachi. The battle was fought between Japanese and American troops over a 8 mi^2 volcanic island 660 *mi* south of Tokyo. It had great value to the Japanese as a base for fighters attacking American bombers on their way to and from bombing missions over Tokyo and other Japanese cities. The American in turn wanted it as a bomber base close to Japan. The American invasion started on February 19, 1945 after a massive, but ineffective, bombardment. (There was an extensive system of caves on the island.) Active fighting stopped on March 26. (The last two Japanese holdouts did not surrender, however, until 1951.) 6,821 Americans were killed, 19,217 were wounded, and 2,648 were hospitalized with combat fatigue. Japanese losses were even more extensive. It is estimated 20,000 Japanese soldiers died out of a total force of 21,000 soldiers. There were a total of 1,083 Japanese prisoners.

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[2] I. David, “Lanchester modeling and the biblical account of the battles of Gibeah,” *Naval Research Logistics*, 42, pp. 579-584 (1995).

[3] J. Bracken, “Lanchester models of the Ardennes campaign,” *Naval Research Logistics*, 42, pp. 559-577 (1995).

[4] R. D. Fricker, “Attrition models of the Ardennes campaign,” *Naval Research Logistics*, 45, pp. 1-22 (1998).

[5] D. S. Hartley and R. L. Helmbold, “Validating Lanchester's square law and other attrition models,” *Naval Research Logistics*, 42, pp. 609-633 (1995).

[6] J. H. Engel, “A verification of Lanchester's law,” *Operations Research*, 2, pp. 163-171 (1954).

The most significant difference between the American forces and the Japanese forces is that the Japanese were isolated and did not receive any reinforcements. The simplest model that might reasonably fit this battle is

$$\frac{dA}{dt} = -b\,J + P(t)\,, \tag{5.32a}$$

$$\frac{dJ}{dt} = -c\,A\,, \tag{5.32b}$$

with the initial conditions

$$A(0) = 0 \tag{5.33a}$$

$$J(0) = J_0 = 21,000\,, \tag{5.33b}$$

where $A(t)$ is the strength of the American forces and $J(t)$ is the strength of the Japanese forces. This system is nonhomogeneous; this makes its treatment more difficult than usual.

The most straightforward approach is to differentiate the second equation

$$\frac{d^2 J}{dt^2} = -c\,\frac{dA}{dt} \tag{5.34}$$

and to then substitute the first equation,

$$\frac{d^2 J}{dt^2} = b\,c\,J - c\,P(t)\,, \tag{5.35}$$

so that

$$\frac{d^2 J}{dt^2} - b\,c\,J = -c\,P(t)\,. \tag{5.36}$$

Once $J(t)$ is known, one can determine $A(t)$ by equation (5.32b).

The solution to equation (5.36) consists of the sum of two linearly independent solutions of the homogeneous and of a particular solution of the nonhomogeneous equation,

$$J(t) = c_1\,J_1(t) + c_2\,J_2(t) + J_p(t)\,. \tag{5.37}$$

The two solutions of the homogeneous equation are pretty straightforward; they can be written as either exponentials or hyperbolic sines and cosines. The particular solution is more difficult. One must use the method of variation of parameters, wherein one assumes a particular solution of the form

$$J_p(t) \;=\; v_1(t)\, J_1(t) + v_2(t)\, J_2(t)\ . \tag{5.38}$$

One can then show that $v_1(t)$ and $v_2(t)$ satisfy

$$v_1(t) \;=\; \int_0^t \frac{c\, P(t)\, J_2(t)}{W[J_1,\, J_2]}\; dt\ , \tag{5.39a}$$

$$v_2(t) \;=\; \int_0^t \frac{-\, c\, P(t)\, J_1(t)}{W[J_1,\, J_2]}\; dt\ , \tag{5.39b}$$

where

$$W[J_1,\, J_2] \;=\; \begin{vmatrix} J_1 & J_2 \\ \dot{J}_1 & \dot{J}_2 \end{vmatrix} \tag{5.40}$$

is the Wronskian. The simplest expression of the solution takes the form

$$A(t) \;=\; -\,\gamma\, J_0 \sinh \beta t + \int_0^t \cosh \beta(t-s)\, P(s)\; ds\ , \tag{5.41a}$$

$$J(t) \;=\; J_0 \cosh \beta t - \frac{1}{\gamma} \int_0^t \sinh \beta(t-s)\, P(s)\; ds\ ,$$

where

$$\beta \;\equiv\; \sqrt{bc}\ , \tag{5.42}$$

and

$$\gamma \;\equiv\; \sqrt{\frac{b}{c}}\ . \tag{5.43}$$

The rate of American reinforcement is known:

$$P(t) \;=\; \begin{cases} 0 \le t < 1 & 0 \le t < 1 \\ 0, & 1 \le t < 2 \\ 6,000, & 2 \le t < 3 \\ 0, & 3 \le t < 5 \\ 13,000, & 5 \le t < 6 \\ 0, & 6 \le t < 36\ . \end{cases}$$

The parameter c may be estimated from equation (5.32b) as

$$J(36) - J_0 = -c \int_0^{36} A_{active}(t)\, dt \tag{5.44}$$

or

(5.45)

The parameter *b* may be estimated in a similar manner. In class, I showed a comparison between the actual and the theoretical troop that appeared in Engel's paper. The agreement was quite good.